KOSHI TENGSIZLIGINING GEOMETRIYADA UCHRAYDIGAN EKSTREMAL TENGSIZLIKLARGA TATBIQI

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Ushbu maqolada olimpiadachi o‘quvchilar sevib ishlatiladigan Koshi tengsizligining geometriyada uchraydigan ekstremal tengsizliklarga tatbiqi keltirilgan. Tengsizliklar yordamida ekstremal masalalarni yechish namunalari keltirilgan. Bundan tashqari Koshi tengsizligiga oid masalalar ham to`plangan. Umid qilamizki, bu maqola o‘quvchi yoshlarimizni layoqati, qobilyati, iqtidorni aniqlash, ochish va ularning rivojlanishi uchun imkoniyatlar yaratib beradi.

1-masala. Agar ΔABC da $p=a+b+c$ va $S-$ yuzasi bo‘lsa, u holda quyidagi tengsizlikni isbotlang.

$$a^2+b^2+c^2\geq\frac{p^2}{3}$$

Isbot. $n=2$ ya’ni $a^2+b^2\geq 2ab$ tengsizlikdan foydalanamiz:

$$\frac{p^2}{3}=\frac{(a+b+c)^2}{3}=\frac{a^2+b^2+c^2+2ab+2ac+2bc}{3}\leq\frac{a^2+b^2+c^2+a^2+b^2+a^2+c^2+b^2+c^2}{3}=$$

$$=\frac{3\left(a^2+b^2+c^2\right)}{3}=a^2+b^2+c^2.$$

2-masala. Agar ΔABC da $p=a+b+c$ va $S-$ yuzasi bo‘lsa, u holda quyidagi tengsizlikni isbotlang.

$$a^3+b^3+c^3\geq\frac{p^3}{9}$$

Isbot. $n=3$ ya’ni $a^3+b^3+c^3\geq 3abc$ tengsizlikdan foydalanamiz:

$$\frac{p^3}{9}=\frac{p}{3}\cdot\frac{p^2}{3}\leq\frac{p}{3}\cdot\left(a^2+b^2+c^2\right)=\frac{(a+b+c)\left(a^2+b^2+c^2\right)}{3}=\frac{a^3+ab^2+ac^2+ba^2+b^3+bc^2+ca^2+cb^2+c^3}{3}=$$

$$\frac{a^3+b^3+c^3+abb+acc+baa+bcc+caa+cbb}{3}\leq$$

$$\leq\frac{a^3+b^3+c^3+\frac{a^3+b^3+b^3}{3}+\frac{a^3+c^3+c^3}{3}+\frac{b^3+a^3+a^3}{3}+\frac{b^3+c^3+c^3}{3}+\frac{c^3+a^3+a^3}{3}+\frac{c^3+b^3+b^3}{3}}{3}=$$

$$=\frac{3\left(a^3+b^3+c^3\right)+6\left(a^3+b^3+c^3\right)}{9}=a^3+b^3+c^3.$$

3-masala. Agar $p-$ uchburchakninig yarim perimetri, $S-$ yuzasi bo‘lsa, u holda

quyidagi tengsizlikni isbotlang.

$$a^3+b^3+c^3 \ge \frac{8\sqrt{3}}{3}Sp$$

Isbot. 2-masalada $p-$ yarim perimetr deb olsak, u holda $a^3+b^3+c^3 \ge \frac{8p^3}{9}$ bo'ladi. Endi quyidagi tengsizlikdan ham foydalanamiz:

$x+y+z \ge 3\sqrt[3]{xyz}$, $x=p-a, y=p-b, z=p-c$ desak,

$$p-a+p-b+p-c \ge 3\sqrt[3]{(p-a)(p-b)(p-c)} \Rightarrow \frac{p^3}{27} \ge (p-a)(p-b)(p-c)$$

$$\frac{8\sqrt{3}}{3}Sp = \frac{8\sqrt{3}}{3}\sqrt{p(p-a)(p-b)(p-c)}\cdot p \le \frac{8\sqrt{3}}{3}\sqrt{p\cdot\frac{p^3}{27}}\cdot p = \frac{8\sqrt{3}}{3}\cdot\frac{p^3}{3\sqrt{3}} = \frac{8p^3}{9} \le a^3+b^3+c^3$$

4-masala. ΔABC da $R-$ tashqi, $r-$ ichki chizilgan aylana radiuslari va $p-$ yarim perimetr bo'lsa, quyidagi tengsizlikni isbotlang.

$$\frac{1}{R^2} \le \frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac} \le \frac{1}{4r^2}$$

Isbot. Quyidagi tengsizlikni isbotlaylik:

$$(p-a)(p-b)(p-c) \le \frac{abc}{8}$$

$$(p-a)(p-b)(p-c) = \frac{2\sqrt{(p-a)(p-b)}\cdot 2\sqrt{(p-a)(p-c)}\cdot 2\sqrt{(p-b)(p-c)}}{8} \le$$

$$\le \frac{(p-a+p-b)(p-a+p-c)(p-b+p-c)}{8} = \frac{abc}{8}$$

Avval tengsizlikni chap qismini isbotlaymiz, ya'ni

$$\frac{1}{R^2} \le \frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}$$

$$\frac{1}{R^2} = \frac{1}{\left(\frac{abc}{4S}\right)^2} = \frac{16S^2}{(abc)^2} = \frac{16p(p-a)(p-b)(p-c)}{(abc)^2} \le \frac{16p\frac{abc}{8}}{(abc)^2} = \frac{2p}{abc} = \frac{a+b+c}{abc} = \frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}.$$

Endi o'ng qismini isbotlaymiz:

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac} \le \frac{1}{4r^2}$$

$$\frac{1}{4r^2} = \frac{1}{4\frac{S^2}{p^2}} = \frac{p^2}{4S^2} = \frac{p^2}{4p(p-a)(p-b)(p-c)} \ge \frac{p^2}{4p\cdot\frac{abc}{8}} = \frac{2p}{abc} = \frac{a+b+c}{abc} = \frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}$$

$$\frac{1}{4r^2} = \frac{1}{4\frac{S^2}{p^2}} = \frac{p^2}{4S^2} = \frac{p^2}{4p(p-a)(p-b)(p-c)} \ge \frac{p^2}{4p\cdot\frac{abc}{8}} = \frac{2p}{abc} = \frac{a+b+c}{abc} = \frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}$$

Bu tengsizlikdan quyidagi natija kelib chiqadi:

Natija. Uchburchakka ichki va tashqi chizilgan aylanalarning radiuslari orasida quyidagi munosabat o'rinli:

$$R \geq 2r$$

Isbot. Oxirgi tengsizlikdan foydalanamiz:

$$\frac{1}{R^2} \leq \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac} \leq \frac{1}{4r^2} \Rightarrow \frac{1}{R^2} \leq \frac{1}{4r^2} \Rightarrow R \geq 2r.$$

5-misol. Agar $a,\ b,\ c$ – sonlar uchburchak tomonlari bo'lib, p – uchburchakning yarim perimetri bo'lsa,

$$(p-a)(p-b)(p-c) \leq \frac{abc}{8}$$

tengsizlik o'rinli bo'lishini isbotlang.

Isbot. Agar $x \geq 0$, $y \geq 0$ bo'lsa, Koshi tengsizligiga ko'ra $2\sqrt{xy} \leq x + y$ tengsizlik o'rinli bo'ladi. Tenglik sharti $x = y$ da bajariladi. Shu tengsizlikdan foydalanamiz.

$$(p-a)(p-b)(p-c)$$
$$= \frac{2\sqrt{(p-a)(p-b)} \cdot 2\sqrt{(p-b)(p-c)} \cdot 2\sqrt{(p-c)(p-a)}}{8}$$
$$\leq \frac{(p-a+p-b)(p-b+p-c)(p-c+p-a)}{8}$$
$$= \frac{(2p-a-b)(2p-b-c)(2p-c-a)}{8} = \frac{abc}{8}$$

Tenglik sharti $p-a = p-b = p-c$, ya'ni $a = b = c$ bo'lganda bajariladi.

6-misol. Ixtiyoriy uchburchak uchun $a^2 + b^2 + c^2 \geq 4\sqrt{3}S$ bo'lishini isbotlang. Bunda uchburchak tomonlari, esa uchburchak yuzasi.

Isbot. Belgilash kiritamiz: $x = p-a$, $y = p-b$, $z = p-c$. Koshi tengsizligiga ko'ra $\sqrt[3]{x \cdot y \cdot z} \leq \frac{x+y+z}{3}$. Bundan

$$\sqrt[3]{(p-a)(p-b)(p-c)} \leq \frac{p-a+p-b+p-c}{3} = \frac{p}{3}$$

ya'ni $(p-a)(p-b)(p-c) \leq \frac{p^3}{27}$ tengsizlikka ega bo'lamiz. Geron formulasiga ko'ra $S = \sqrt{p(p-a)(p-b)(p-c)} \leq \sqrt{p \cdot \frac{p^3}{27}} = \frac{p^2}{3\sqrt{3}} = \frac{(a+b+c)^2}{12\sqrt{3}} = \frac{a^2+b^2+c^2+2ab+2bc+2ac}{12\sqrt{3}}$

tengsizlikka ko'ra $2ab \leq a^2 + b^2$; $2bc \leq b^2 + c^2$; $2ac \leq a^2 + c^2$

Shuning uchun

$S \leq \frac{3(a^2+b^2+c^2)}{12\sqrt{3}} = \frac{a^2+b^2+c^2}{4\sqrt{3}}$.

Demak, $a^2 + b^2 + c^2 \geq 4\sqrt{3}S$ tenglik sharti $a = b = c$ bo'lganda bajariladi.

7-misol. Uzunligi l bo'lgan kesmaning o'zaro perpendikulyar bo'lgan uchta tekisliklarga proeksiyalari a, b, c bo'ladi, $\frac{a+b+c}{l}$ nisbat eng katta qiymat qabul qilishi uchun bu kesma qanday joylashishi kerakligini toping.

Isboti. Bu masalani yechish uchun berilgan tekisliklarni koordinatalar tekisliklari deb qaraymiz va kesma uchlarini $A(x_1,\ y_1, z_1), B(x_2, y_2, z_2)$ bilan belgilaymiz. U holda

$$l = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$a = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$b = \sqrt{(x_2 - x_1)^2 + (z_2 - z_1)^2}$$

$$c = \sqrt{(y_2 - y_1)^2 + (z_2 - z_1)^2}$$

bo`ladi. Bunga ko‘ra $a^2 + b^2 + c^2 = 2l^2$ bo`ladi. Koshi tengsizligiga asosan

$$\frac{a+b+c}{l} = \frac{\sqrt{(a+b+c)^2}}{l} = \frac{\sqrt{a^2+b^2+c^2+2ab+2bc+2ac}}{l}$$

$$\leq \frac{\sqrt{3(a^2+b^2+c^2)}}{l} = \frac{\sqrt{3 \cdot 2l^2}}{l} = \sqrt{6}$$

bo‘ladi. Bu yerda tenglik bajarilishi uchun $a = b = c$ bo`lishi kerak. Demak, $\frac{a+b+c}{l}$ nisbat eng katta qiymatiga erishish uchun berilgan kesma berilgan tekisliklarga bir xil og`ishgan bo`lishi lozim. Nisbatning eng katta qiymati esa $\sqrt{6}$ ga teng.

Foydalanilgan adabiyotlar

1. H.Norjigitov, J.A.Bahramov “ Matematik olimpiada masalalarini yechish uchun qo‘llanma” Toshkent, 2014-yil.147-bet.

2. H.Norjigitov, A.X.Nuraliyev “Matematikadan olimpiada masalalari” Toshkent, 2020-yil. 243-bet.

3. B.I.Abdullayev, J.U.Xujamova, R.A.Sharipov “Matematikadan olimpiada masalalari” Urganch, 2016-yil 60-bet.